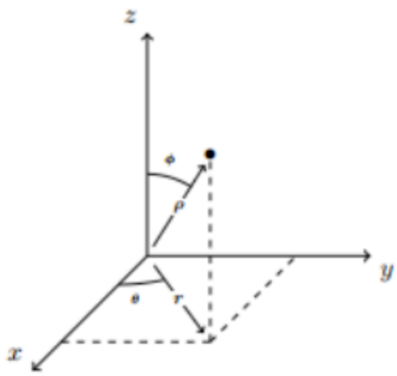
- The relations, still introducing an extra variable r as in polar coordinates (it will be very useful)

$$\begin{cases} x = (\rho\cos\phi)\cos\theta \\ y = (\rho\cos\phi)\sin\theta \\ z = \rho\cos\phi \end{cases} \quad \text{and} \quad \begin{cases} \rho^2 = x^2+y^2+z^2 \\ r = \rho\sin\phi \\ \frac{r}{z} = \tan\phi \end{cases}, \quad \theta\in[0,2\pi], \phi\in[0,\pi]$$



- If $\phi > \frac{\pi}{2}$ then $z<0$, the angle make P lies below the Oxy-plane.

The Jacobian

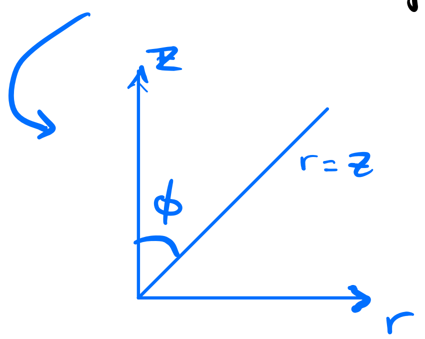
$$\boxed{\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi}$$

Goal.

1. sketch the solid region in (r,z) plane
2. Solving for the bounds.

$$\iiint_V f(x,y,z)\,dV = \int_\theta\int_\phi\int_\rho f(\rho,\phi,\theta)\,\underbrace{\boxed{\rho^2\sin\phi}}_{\text{Jacobian}}\,d\rho\,d\phi\,d\theta$$

Ex 1. Sketch $\phi = \pi/4$ and $\rho = 2\cos\phi$ in (r,z) plane and 3D space.



as $r = \rho\sin\phi$, $z = \rho\cos\phi$ $\Rightarrow \frac{r}{z} = \tan\phi = 1$

$\Rightarrow r = z$

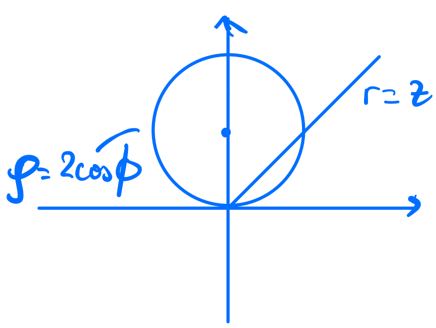
$\rho = 2\cos\phi \Rightarrow \rho^2 = 2\rho\cos\phi$

$x^2+y^2+z^2 = 2z \Rightarrow x^2+y^2+z^2-2z = 0$

$\Rightarrow \underbrace{x^2+y^2}_{r^2} + \underbrace{z^2-2z+1}_{(z-1)^2} = 1$

$r^2 + (z-1)^2 = 1$

↳ circle of radius 1 at $(0,1)$



Remark: Using a sketch in (r,z) plane only if the surface doesnot depend on θ.

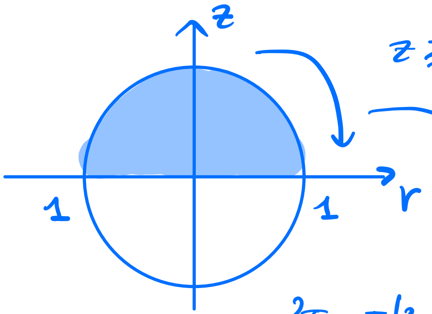
Ex2.

$f(x,y,z) = z$ is the height of (x,y,z)

Calculate the average height of a point inside the hemisphere $\underbrace{x^2 + y^2 + z^2 = 1}$ with $z \geq 0$.

$\rho = 1$

$r^2 + z^2 = 1$



$z \geq 0$

$0 \leq \phi \leq \frac{\pi}{2}$, $0 \leq \theta \leq 2\pi$

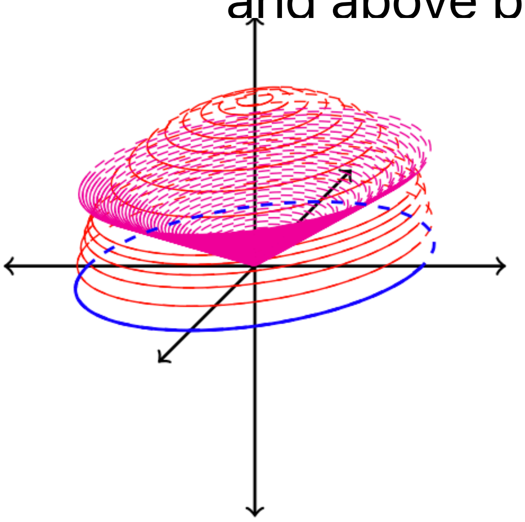
$$\text{Vol} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{1} 1 \, \underbrace{\rho^2 \sin\phi}_{\text{Jacobian}} \, d\rho \, d\phi \, d\theta$$

$$\text{Answer} = \frac{1}{\text{Vol}} \int_0^{2\pi} \int_0^{\pi/2} \int_0^{1} (\underbrace{\rho\cos\phi}_{z}) \, (\underbrace{\rho^2 \sin\phi}_{\text{Jacobian}}) \, d\rho \, d\phi \, d\theta$$

Ex 3. Use spherical integration to find the volume of the solid bounded below by the cone $z = \sqrt{\frac{x^2+y^2}{3}}$

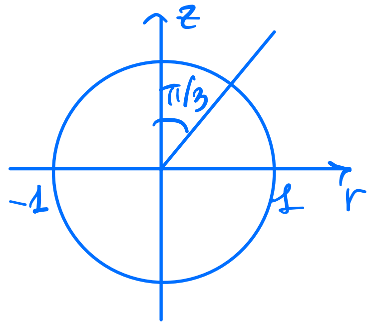
and above by the hemisphere

$z = \sqrt{1 - x^2 - y^2}$



$z = \sqrt{1-x^2-y^2} \Rightarrow z^2 = 1-x^2-y^2 \Rightarrow x^2+y^2+z^2 = 1$

$\begin{cases} \rho = 1 \\ \text{or} \quad r^2+z^2 = 1 \end{cases}$



$z = \sqrt{\dfrac{x^2+y^2}{3}} \Rightarrow \rho\cos\phi = \dfrac{\rho\sin\phi}{\sqrt{3}}$

$= \dfrac{r}{\sqrt{3}}$

$\tan\phi = \sqrt{3}$

(note: $r = \rho\sin\phi$)

$\phi = \dfrac{\pi}{3}$

$0 \le \phi \le \dfrac{\pi}{3}, \quad 0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} 1 \cdot \underbrace{\boxed{\rho^2\sin\phi}}_{\text{Jacobian}} \, d\rho\, d\phi\, d\theta$$

Ex 4.

Note: $a^2 - 2ab + b^2 = (a-b)^2$

Find the volume of the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2$, $z \geq 0$.

$\rho = \cos\phi$

$\Rightarrow \rho^2 = \rho\cos\phi$

$\Rightarrow x^2+y^2+z^2 = z$

$\Rightarrow x^2+y^2+z^2-z = 0$

$\Rightarrow x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow r^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4}$

circle at $(0, \frac{1}{2})$, radius $\frac{1}{2}$

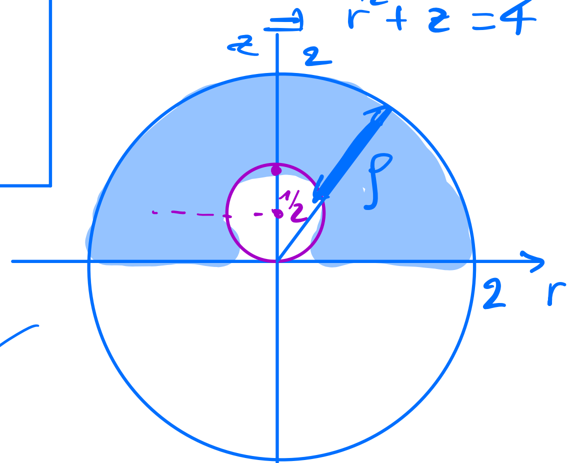
Complete the square

$z^2 - z = \underbrace{z^2 - 2z\cdot\frac{1}{2} + \frac{1}{4}}_{\left(z-\frac{1}{2}\right)^2} - \frac{1}{4}$

$\left(z-\frac{1}{2}\right)^2 - \frac{1}{4}$

$\rho = 2 \Rightarrow x^2+y^2+z^2 = 4$

$\Rightarrow r^2+z^2 = 4$



$\cos\phi \le \rho \le 2$

$0 \le \phi \le \pi/2 \quad (z \ge 0$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^{2} 1 \cdot \underbrace{\boxed{\rho^2\sin\phi}}_{\text{Jacobian}} \, d\rho\, d\phi\, d\theta$$

$\boxed{\text{Ex 5.}}$ (Extra problem)

Find the volume of the solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2+y^2}$

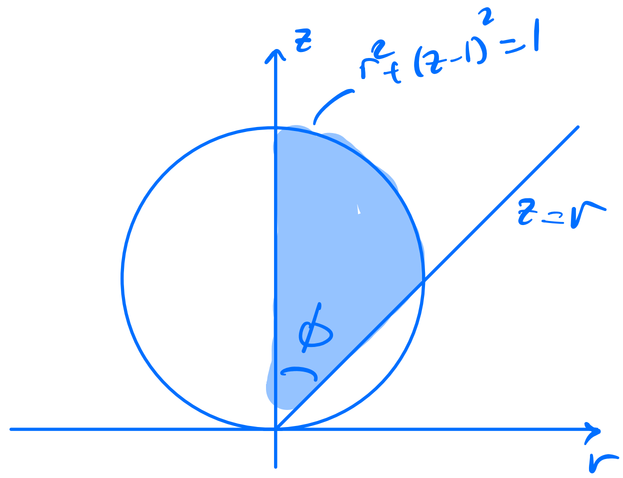
$\rho = 2\cos\phi$

$\rho^2 = 2\rho\cos\phi$

$x^2+y^2+z^2 = 2z \Rightarrow \underbrace{x^2+y^2}_{r^2} + (z-1)^2 = 1$

Circle $r^2 + (z-1)^2 = 1$

$\hookrightarrow$ center $(0,1)$, radius 1



$z = \sqrt{x^2+y^2} \Rightarrow z = r$

$\hookrightarrow \rho\cos\phi = \rho\sin\phi \Rightarrow \tan\phi = 1 \Rightarrow \phi = \pi/4$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} 1 \cdot \underbrace{\boxed{\rho^2\sin\phi}}_{\text{Jacobian}} \, d\rho \, d\phi \, d\theta$$

$\rightarrow \rho^2$

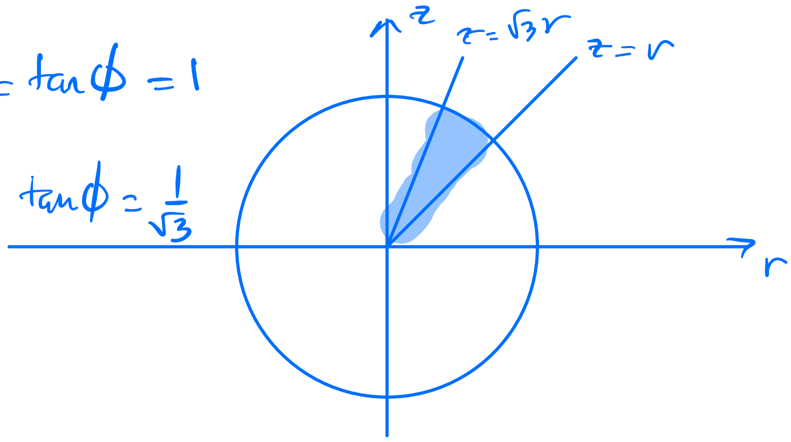
Ex. 6 Evaluate the integral $\iiint_E (x^2+y^2+z^2)\, dV$ where E is the region between two half-cones $z=\sqrt{x^2+y^2}$ and $z=\sqrt{3x^2+3y^2}$ and bounded by the hemisphere $x^2+y^2+z^2=9$.

$z=\sqrt{x^2+y^2} \Rightarrow z=r \Rightarrow \frac{r}{z}=\tan\phi=1$

$z=\sqrt{3(x^2+y^2)} \Rightarrow z=\sqrt{3}\,r \Rightarrow \frac{r}{z}=\tan\phi=\frac{1}{\sqrt{3}}$

$z^2+x^2+y^2=9 \Rightarrow z^2+r^2=9$

$\hookrightarrow \rho=3$



thus $\tan\phi=1 \Rightarrow \phi=\frac{\pi}{4}$

$\tan\phi=\frac{1}{\sqrt{3}} \Rightarrow \phi=\frac{\pi}{6}$

$\Rightarrow \frac{\pi}{6}\le\phi\le\frac{\pi}{4}$

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\int_0^{3} \rho^2\cdot \underbrace{\boxed{\rho^2\sin\phi}}_{\text{Jacobian}}\, d\rho\, d\phi\, d\theta$$